# Hyperinvariant subspaces for $n$-normal operators

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**1. Introduction.** Let $\mathfrak{H}$ be a complex Hilbert space and let $\mathscr{L}(\mathfrak{H})$ be the algebra of all bounded linear operators on $\mathfrak{H}$. (In what follows, all Hilbert spaces will be complex, and all operators under discussion will be bounded and linear.) A closed subspace $\mathfrak{M}\subset\mathfrak{H}$ is said to be *hyperinvariant* for an operator $A$ in $\mathscr{L}(\mathfrak{H})$ if it is nontrivial and is invariant for every operator which commutes with $A$; that is, if $\mathfrak{M}$ is distinct from $\{0\}$ and $\mathfrak{H}$ and $B(\mathfrak{M})\subset\mathfrak{M}$ for every operator $B$ in $\mathscr{L}(\mathfrak{H})$ satisfying $AB=BA$.

The notion of hyperinvariant subspace was introduced by SZ.-NAGY and FOIAŞ (under the name "ultrainvariant") [10]; these authors and later DOUGLAS and PEARCY [3], [4] characterized the hyperinvariant subspaces of certain types of operators and gave a number of sufficient conditions for an invariant subspace to be hyperinvariant.

The principal purpose of this paper is to show (Theorem 5. 3) that every operator which is *n-normal*, in a sense to be defined below, has a hyperinvariant subspace. Let $\mathfrak{H}^n = \mathfrak{H}\oplus\mathfrak{H}\oplus\cdots\oplus\mathfrak{H}$ denote the orthogonal sum of $n$ copies of the Hilbert space $\mathfrak{H}$. One knows that every operator in $\mathscr{L}(\mathfrak{H}^n)$ can be written as an $n\times n$ matrix $(A_{ij})_{i,j=1}^n$ where each $A_{ij}$ $(1\leqq i,j\leqq n)$ belongs to $\mathscr{L}(\mathfrak{H})$. An operator $B$ on a Hilber space $\mathfrak{K}$ is said to be $n$-normal if there is a Hilbert space $\mathfrak{H}$ and $n^2$ mutually commuting normal operators $A_{ij}$ $(1\leqq i,j\leqq n)$ acting on $\mathfrak{H}$ such that $\mathfrak{K}=\mathfrak{H}^n$ and $B=(A_{ij})_{i,j=1}^n$.

The class of $n$-normal operators may be defined equivalently using the concept of *von Neumann algebras,* i.e. of weakly closed, self-adjoint algebras of operators on Hilbert space, containing the identity operator. If $\mathscr{A}$ is an abelian von Neumann algebra acting on $\mathfrak{H}$ then $M_n(\mathscr{A})$ will denote the von Neumann algebra consisting of all $n\times n$ matrices with entries from $\mathscr{A}$ acting on $\mathfrak{H}^n$ in the usual fashion. It is immediate that an operator $A$ on a Hilbert space is $n$-normal if and only if there

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exists an abelian von Neumann algebra $\mathscr{A}$ such that $A$ belongs to $M_n(\mathscr{A})$. (It is worth noting that there are operators which are $n$-normal in the sense of [7] but are not $n$-normal in our sense. (See § 6.)

The proof that every $n$-normal operator has a hyperinvariant subspace is accomplished in two steps. First we show that if $A$ and $B$ are quasi-similar operators and if $B$ has a hyperinvariant subspace, then so does $A$. (See Section 2.) Next, the algebra $M_n(\mathscr{A})$ is identified with an algebra of continuous matrix-valued functions on an extremally disconnected compact Hausdorff space. Using this identification and the techniques developed in [1], [2], and [8], every operator is shown to be quasi-similar to an operator $J$ in $M_n(\mathscr{A})$ which is in "Jordan form". These operators are easily seen to be spectral operators in the sense of Dunford, and thus to have hyperinvariant subspaces [5].

**2. Quasi-similarity of operators.** If $A$ and $B$ are operators on the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ respectively, then $A$ and $B$ are said to be *quasi-similar* if there are bounded linear operators $R\colon \mathfrak{K}\to\mathfrak{H}$ and $S\colon \mathfrak{H}\to\mathfrak{K}$ which satisfy the following conditions:

1. $SA=BS$ and $AR=RB$.
2. $R$ and $S$ have zero kernels and dense ranges.

In [11], Sz.-Nagy and Foiaş prove that a quasi-similarity between an operator $A$ and a unitary operator $U$ induces a one to one, order preserving map of the lattice of hyperinvariant subspaces of $U$ into that of $A$. Examination of their argument also yields the following:

Theorem 2. 1. *If $A$ and $B$ are quasi-similar operators on Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ respectively, and if $B$ has a hyperinvariant subspace, then so does $A$.*

Proof. Let $R$ and $S$ be the operators which invoke the quasi-similarity and let $\mathfrak{M}$ be the subspace of $\mathfrak{K}$ which is hyperinvariant for $B$. Define:

$$a(\mathfrak{M})=\overline{R(\mathfrak{M})} \quad\text{and}\quad b(\mathfrak{M})=\{f\in\mathfrak{H}\mid S(f)\in\mathfrak{M}\}.$$

Because $\mathfrak{M}\neq\mathfrak{K}$ and $S$ has dense range, $b(\mathfrak{M})\neq\mathfrak{H}$. Also, $a(\mathfrak{M})\neq\{0\}$ since $R$ is one to one and $\mathfrak{M}\neq\{0\}$.

If $T$ is an operator on $\mathfrak{H}$ which commutes with $A$, then

$$BSTR=SATR=STAR=STRB.$$

Thus $STR$ commutes with $B$ and therefore $STR(\mathfrak{M})\subset\mathfrak{M}$. It follows that $Ta(\mathfrak{M})=\overline{TR(\mathfrak{M})}\subset b(\mathfrak{M})$. Define $q(\mathfrak{M})$ to be the smallest closed subspace of $\mathfrak{H}$ which contains $Ta(\mathfrak{M})$ for every $T$ in $\mathscr{L}(\mathfrak{H})$ that commutes with $A$. Then $\{0\}\neq a(\mathfrak{M})\subset q(\mathfrak{M})\subset b(\mathfrak{M})\neq\mathfrak{H}$ and $q(\mathfrak{M})$ is hyperinvariant for $A$.

Examination of the above argument shows that if $B$ has two distinct hyperinvariant subspaces $\mathfrak{M}$ and $\mathfrak{N}$ with $\mathfrak{N} \subset \mathfrak{M}$, then $q(\mathfrak{N}) \subset q(\mathfrak{M})$. However, the following can be proved using the argument of [10, Proposition II. 5. 1]:

*Suppose $A$, $B$ and $q$ are as in Theorem 2. 1 and suppose $B$ is normal. If $\mathfrak{M}$ and $\mathfrak{N}$ are distinct hyperinvariant subspaces for $B$, then $q(\mathfrak{M}) \neq q(\mathfrak{N})$.*

**3. Quasi-similarity in $M_n(X)$.** In what follows, $X$ will denote an extremally disconnected compact Hausdorff space; i.e., the closure of every open subset of $X$ is compact and open. Such topological spaces are called Stonian spaces, and they arise naturally as the maximal ideal spaces of von Neumann algebras. They may be characterized as those compact Hausdorff spaces whose lattice of real continuous functions is complete [9], and they are known to have a base of compact open sets. Let $M_n$ denote the set of all $n \times n$ complex matrices viewed as the algebra of operators on $n$-dimensional complex Hilbert space. By $M_n(X)$ we will mean the algebra of all continuous $M_n$-valued functions $A(\cdot)$ on $X$ with operations defined pointwise and

$$\|A(\cdot)\| = \sup_{x \in X} \|A(x)\|.$$

It is easy to verify that $M_n(X)$ is a $C^*$-algebra.

Elements of $M_n(X)$ may also be viewed as $n \times n$ matrices with entries in $C(X)$, the algebra of all continuous, complex valued function defined on $X$. The algebras $M_n(X)$ have been studied extensively by PEARCY and DECKARD in [1], [2], and [8].

A matrix $R(\cdot)$ in $M_n(X)$ is said to be *quasi-invertible* if whenever $D(\cdot)$ in $M_n(X)$ satisfies $D(\cdot)R(\cdot)=0$ or $R(\cdot)D(\cdot)=0$ then $D(\cdot)=0$. Matrices $A(\cdot)$ and $B(\cdot)$ are said to be *quasi-similar in* $M_n(X)$ if there exist quasi-invertible matrices $R(\cdot)$ and $S(\cdot)$ in $M_n(X)$ which satisfy

$$S(\cdot)A(\cdot)=B(\cdot)S(\cdot) \quad \text{and} \quad A(\cdot)R(\cdot)=R(\cdot)B(\cdot).$$

Lemma 3. 1. *Suppose that $A(\cdot)$ and $B(\cdot)$ are matrices in $M_n(X)$ and that for each $x$ in a dense subset $D$ of $X$, $A(x)$ is similar to $B(x)$. Then for each open subset $U$ of $X$ there is a compact open set $V \subset U$ and an invertible element $S(\cdot)$ in $M_n(V)$ such that $S(x)A(x)=B(x)S(x)$ for each $x$ in $V$.*

Proof. Consider the system $L$ of homogeneous linear equations with coefficients in $C(X)$ which corresponds to the matrix equation $S(\cdot)A(\cdot)=B(\cdot)S(\cdot)$:

$$\text{(L)} \qquad \begin{array}{c} c_{11}s_1+c_{12}s_2+\cdots+c_{1m}s_m=0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ c_{m1}s_1+c_{m2}s_2+\cdots+c_{mm}s_m=0, \end{array}$$

where $m=n^2$ and the unknown functions $s_i$ represent the entries of $S(\cdot)$ in some prescribed order. For $x$ in $X$, let $L(x)$ be the corresponding system of scalar equations.

Choose an $x_0$ in $U$ so that $v_0$, the rank of $L(x_0)$, is maximal for $x$ in $U$. There is a $v_0 \times v_0$ minor $N(x_0)$ of the matrix of coefficients of $L(x_0)$ such that $\det N(x_0) \neq 0$. Consequently, $\det N(\cdot)$ does not vanish on a compact open neighborhood $V_1$ of $x_0$, $V_1 \subset U$. By the hypothesis, there is an $x_1$ in $D \cap V_1$ and an invertible complex matrix $T_{x_1}$ satisfying $T_{x_1}A(x_1) = B(x_1)T_{x_1}$. If $u_1, \ldots, u_m$ are the entires in $T_{x_1}$, then $(u_1 \ldots, u_m)$ is a solution to $L(x_1)$. For $i$ not affiliated with $N(\cdot)$, let $s_i \equiv u_i$ on $V_1$. For the $v_0$ values of $i$ affiliated with $N(\cdot)$, use the already assigned $s_j$ and Cramer's rule to define $s_i$. Since the $c_{ij}$ are continuous and $\det N(x) \neq 0$ for $x$ in $V_1$, the $s_i$ are continuous and satisfy $L$. Thus if $S(\cdot)$ is the matrix in $M_n(V_1)$ with the $s_i$ in the appropriate positions, then $S(x)A(x) = B(x)S(x)$ for each $x$ in $X$.

Since $S(x_1) = T_{x_1}$ is invertible, and the set of invertible matrices is open, there is a compact open set $V \subseteq V_1$ such that the restriction of $S(\cdot)$ to $V$ is invertible in $M_n(V)$.

Theorem 3.2. *If $A(\cdot)$ and $B(\cdot)$ satisfy the conditions of Lemma* 3.1, *then $A(\cdot)$ and $B(\cdot)$ are quasi-similar in $M_n(X)$.*

Proof. Let $\mathcal{F}$ denote the collection of all families $\{U_\alpha\}_{\alpha \in I}$ of disjoint compact open subsets of $X$ such that for each $\alpha$ in $I$ there is an $S_\alpha(\cdot)$ in $M_n(U_\alpha)$ which is invertible in $M_n(U_\alpha)$ and satisfies $S_\alpha(x)A(x) = B(x)S_\alpha(x)$ for each $x$ in $U_\alpha$. Order $\mathcal{F}$ by inclusion and use Zorn's lemma to obtain a maximal family $\{U_\alpha\}_{\alpha \in I}$ in $\mathcal{F}$.

If $Y = \bigcup_{\alpha \in I} U_\alpha$ is not dense in $X$, then by Lemma 3.1 there is a compact open set $V \subset X - Y$ and an $S(\cdot)$ in $M_n(V)$ which affects a similarity between the restrictions of $A(\cdot)$ and $B(\cdot)$ to $V$. This contradicts the maximality of $\{U_\alpha\}_{\alpha \in I}$.

By Lemma 2.1 of [1], there are matrices $S(\cdot)$ and $R(\cdot)$ in $M_n(X)$ which extend each

$$\frac{1}{\|S_\alpha(\cdot)\|} S_\alpha(\cdot) \quad \text{and} \quad \frac{1}{\|S_\alpha^{-1}(\cdot)\|} S_\alpha^{-1}(\cdot)$$

respectively. These matrices satisfy the equalities

$$S(\cdot)A(\cdot) = B(\cdot)S(\cdot) \quad \text{and} \quad A(\cdot)R(\cdot) = R(\cdot)B(\cdot).$$

It remains to show that $R(\cdot)$ and $S(\cdot)$ are quasi-invertible. Suppose $C(\cdot)$ is a matrix in $M_n(X)$ and $C(\cdot)R(\cdot) = 0$; that is, $C(x)R(x) = 0$ for each $x$ in $X$. For each $x$ in the dense subset $Y$ of $X$, $R(x)$ is invertible and so $C(x) = 0$. It follows that $C(\cdot) = 0$. The other three implications are easily proved in the same way.

**4. Jordan forms in $M_n(X)$.** In [2], Deckard and Pearcy exhibit a Stonian space $X$ and a matrix $A(\cdot)$ in $M_2(X)$ for which there is no $J(\cdot)$ in $M_2(X)$ which is similar to $A(\cdot)$ and is such that $J(x)$ is in Jordan form for each $x$ in $X$. If the condition of similarity is relaxed to quasisimilarity, then such Jordan forms always exist. This is shown via the following lemmas.

Lemma 4.1. *If $\varphi_1, \ldots, \varphi_n$ are in $C(X)$, where $X$ is a Stonian space, and if $U$ is a non-empty, open subset of $X$, then there is a non-empty, compact, open set $V \subset U$ such that for each $i$ and $j$ ($1 \leqq i, j \leqq n$) either $\varphi_i(x) = \varphi_j(x)$ for all $x$ in $V$, or $\varphi_i(x) \neq \varphi_j(x)$ for all $x$ in $V$.*

Proof. Pick $x_0$ in $U$ so that the number of distinct values $\varphi_i(x_0)$ is maximal; assume these values are $\varphi_{i_1}(x_0), \ldots, \varphi_{i_l}(x_0)$. There is an open neighborhood $U_0$ of $x_0$, $U_0 \subset U$, such that $\varphi_{i_j}(x) \neq \varphi_{i_k}(x)$ for $j \neq k$ and $x$ in $U_0$. For each $i$, $\varphi_i(x_0) = \varphi_{i_j}(x_0)$ for some $j$ and hence $\varphi_i(x_0) \neq \varphi_{i_k}(x_0)$ for $k \neq j$. Therefore $\varphi_i(x) \neq \varphi_{i_k}(x)$ for each $k$, $k \neq j$ and for each $x$ in some compact open neighborhood $V_i$ of $x_0$, $V_i \subset U_0$. Consequently, $\varphi_i(x) = \varphi_{i_j}(x)$ for each $x$ in $V_i$ and $V = \bigcap_{i=1}^{n} V_i$ is the desired set.

Lemma 4.2. *If $U$ is non-empty subset of the Stonian space $X$, and if $B(\cdot)$ is in $M_n(X)$, then there is a non-empty compact open set $V \subset U$ on which the rank of $B(\cdot)$ is constant.*

Proof. Choose an $x_0$ in $U$ so that $r_0$, the rank of $B(x_0)$, is maximal for $x$ in $U$. There is an $r_0 \times r_0$ minor $M(x_0)$ of $B(x_0)$ with $\det M(x_0) \neq 0$, and hence $\det M(\cdot)$ does not vanish in some compact open neighborhood $V$ of $x_0$, $V \subset U$. It follows that the rank of $B(\cdot) \equiv r_0$ on $V$.

Suppose that $A$ and $A'$ are $n \times n$ scalar matrices in Jordan form, having the single eigenvalues $\lambda$ and $\lambda'$ respectively. More explicitly, suppose $A = \sum_{i=1}^{k} \oplus A_i$ where $A_i$ is an $s_i \times s_i$ Jordan block matrix for $\lambda$ and $s_{i-1} \leqq s_i$ for $1 < i \leqq k$. Similarly, $A' = \sum_{i=1}^{l} \oplus A_i'$ where $A_i'$ is an $s_i' \times s_i'$ Jordan block for $\lambda'$ and the $A_i'$ are arranged in order of decreasing size. (A Jordan block for $\lambda$ is a square matrix with each entry on the main diagonal equal to $\lambda$, with ones on the diagonal above the main diagonal, and with zeros in all other positions. A finite scalar matrix is in Jordan form if it is the direct sum of Jordan blocks.)

Lemma 4.3. *If, in the notation established above,* $\mathrm{Rank}(A-\lambda)^r = \mathrm{Rank}\,(A'-\lambda')^r$ *for each $r \leqq \max\{s_1, s_1'\}$, then $k = l$ and $s_i = s_i'$ for each $i$, $1 \leqq i \leqq k$.*

Proof. This lemma is proved by induction on $\max\{s_1, s_1'\}$. Since

$$0 = \mathrm{Rank}\,(A-\lambda)^{s_1} = \mathrm{Rank}\,(A'-\lambda')^{s_1}$$

and $A'-\lambda'$ is nilpotent of index $s_1'$, $s_1 \geqq s_1'$. Similarly, $s_1' \geqq s_1$ and so $s_1 = s_1'$. Beacuse $\mathrm{Rank}\,(A-\lambda)^{s_1-1} = \mathrm{Rank}\,(A'-\lambda')^{s_1-1}$, the number of $s_i$ equal to $s_1$ is the same as the number of $s_j'$ equal to $s_1$. Consequently, if $m$ is this number then the matrices

$$B = \mathrm{diag}\,(A_{m+1}, \ldots, A_k) \quad \text{and} \quad B' = \mathrm{diag}\,(A_{m+1}', \ldots, A_l')$$

satisfy the hypothesis of the lemma and $\max\{s_{m+1}, s'_{m+1}\} < s_1$. Therefore, by the induction hypothesis, $k-m = l-m$ and $s_i = s'_i$ for $m+1 \leqq i \leqq k$.

In [1], DECKARD and PEARCY prove that if $X$ is a Stonian space and if $\varrho(\lambda, x) = \lambda^n + a_{n-1}(x)\lambda^{n-1} + \dots + a_2(x)\lambda + a_0(x)$ is a monic polynomial with coefficients in $C(X)$, then there is a function $\varphi$ in $C(X)$ such that $\varrho(\varphi(x), x) = 0$ for every $x$ in $X$. It follows that all such polynomials can be written in the form $\varrho(\lambda, x) = \prod_{i=n}^{n} (\lambda - \varphi_i(x))$ where the functions $\varphi_i$ $(1 \leqq i \leqq n)$ are in $C(X)$. Of particular interest in this paper is the case when $\varrho(\lambda, x)$ is the characteristic polynomial of some matrix $A(\cdot)$ in $M_n(X)$:

$$\varrho(\lambda, x) = \det[\lambda I - A(x)].$$

In this case, $\varrho(A(x), x) = 0$ for each $x$ in $X$.

Lemma 4. 4. *If $U$ is a non-empty open subset of the Stonian space $X$, and if $A(\cdot)$ is a matrix in $M_n(X)$, then there is a non-empty, compact, open set $V$ contained in $U$ and a matrix $J(\cdot)$ in $M_n(V)$ such that $J(x)$ is in Jordan form and is similar to $A(x)$ for each $x$ in $V$.*

Proof. By virtue of Lemmas 4. 1 and 4. 2 there is a compact open set $V \subset U$ which satisfies the following conditions:

1. The restriction to $V$ of the characteristic polynomial $\varrho(\lambda, \cdot)$ of $A(\cdot)$ can be factored as

$$\varrho(\lambda, \cdot) = \prod_{i=j}^{k} (\lambda - \varphi_i)^{r_i}$$

where for $i \neq j$, $\varphi_i(x) \neq \varphi_j(x)$ for each $x$ in $V$.

2. For every set of positive integers

$$\{s_i : 1 \leqq i \leqq k,\ 0 \leqq s_i \leqq r\},$$

the matrix $\prod_{i=1}^{k} (A(\cdot) - \varphi_i)^{S_i}$ has constant rank on $V$.

For $x$ in $X$, let $J(x) = \operatorname{diag}(J_x^1, J_x^2, \dots, J_x^k)$ be the matrix similar to $A(x)$ where $J_x^i$ is a $t_x^i \times t_x^i$ matrix in Jordan form with a single eigenvalue $\varphi_i(x)$ and the Jordan blocks of $J_x^i$ are arranged in order of decreasing size. If $x$ and $y$ are in $V$, then for $1 \leqq i \leqq k$, $J_x^i$ and $J_y^i$ satisfy the conditions of Lemma 4. 3. In fact,

$$t_x^i = n - \operatorname{Rank}(J(x) - \varphi_i(x))^{r_i} = n - \operatorname{Rank}(A(x) - \varphi_i(x))^{r_i} =$$
$$= n - \operatorname{Rank}(A(y) - \varphi_i(y))^{r_i} = n - \operatorname{Rank}(J(x) - \varphi_i(y)) = t_y^i,$$

and, for $s_i \leqq t_x^i$,

$$\operatorname{Rank}(J_x^i - \varphi_i(x))^{S_i} = \operatorname{Rank}(J(x) - \varphi_i(x))^{S_i} - (n - t_x^i) =$$
$$= \operatorname{Rank}(J(x) - \varphi_i(x))^{S_i} - (n - t_y^i) = \operatorname{Rank}(J_y^i - \varphi_i(y))^{S_i}.$$

Consequently, $J_x^i$ and $J_y^i$ differ only along the main diagonal, and hence the same is true of $J(x)$ and $J(y)$.

It is now easy to see that the matrix valued function $J(\cdot)$ is continuous; in fact, the only entries along its main diagonal are the functions $\varphi_i$, and the entries in all other positions are constant functions. Thus $J(\cdot)$ is in $M_n(V)$, and the proof is complete.

Theorem 4. 5. *If $X$ is a Stonian space and $A(\cdot)$ is a matrix in $M_n(X)$, then there is a $J(\cdot)$ in $M_n(X)$ such that $J(x)$ is in Jordan form for each $x$ in $X$ and such that for each $x$ in a dense subset $D$ of $X$, $J(x)$ is similar to $A(x)$.*

Proof. Let $\mathscr{F}$ be the family of all collections $\{U_\alpha\}_{\alpha\in I}$ of non-empty disjoint, compact open subsets of $X$ where for each $\alpha$ in $I$ there is a $J_\alpha(\cdot)$ in $M_n(U_\alpha)$ such that $J_\alpha(x)$ is in Jordan form and similar to $A(x)$ for each $x$ in $U_\alpha$. Order $\mathscr{F}$ by inclusion and use Zorn's lemma to obtain a maximal family $\{U_\alpha\}_{\alpha\in I}$ in $\mathscr{F}$. If $D = \bigcup_{\alpha\in I} U_\alpha$ is not dense in $X$, then by Lemma 4. 4, there is a non-empty compact open set $V$ contained in $X-\bar{D}$ and a $J(\cdot)$ in $M_n(V)$ which pointwise is in Jordan form and is similar to $A(\cdot)$. But this contradicts the maximality of $\{U_\alpha\}_{\alpha\in I}$. Therefore $D$ is dense in $X$. For each $\alpha$ in $I$ and $x$ in $U_\alpha$, the entries in $J_\alpha(x)$ are all bounded by $\max\{\|\varphi\|, \ldots, \|\varphi_n\|, 1\}$; therefore, by Lemma 2. 1 of [1], there is a $J(\cdot)$ in $M_n(X)$ which extends each $J_\alpha(\cdot)$.

It remains to show that for $x_0$ in $X-D$, $J(x_0)$ is in Jordan form. Suppose $J(\cdot)=(J_{ij})_{i,j=1}^n$ where each $J_{ij}$ is in $C(X)$; then for $j\neq i, i+1$, $J_{ij}$ is the zero function and for $j=i+1$, $J_{ij}(x_0)$ is either zero or one. Suppose $J_{i,i+1}(x_0)=1$, then if $\langle x_\beta\rangle$ is any net in $D$ which converges to $x_0$, $J_{i,i+1}(x_\beta)$ converges to 1. Since for each $\beta$, $J_{i,i+1}(x_\beta)$ is either zero or one, the net $\langle J_{i,i+1}(x)_\beta\rangle_\beta$ is eventually identically equal to one. But each $J(x_\beta)$ is in Jordan form, so eventually, $J_{ii}(x_\beta)=J_{i+1,i+1}(x_\beta)$ and hence $J_{i,i}(x_0)=J_{i+1,i+1}(x_0)$. Therefore $J(x_0)$ is in Jordan form.

Using Theorems 3. 2 and 4. 5, the following is obtained:

Theorem 4. 6. *If $X$ is a Stonian space and $A(\cdot)$ is a matrix in $M_n(X)$, then there is a $J(\cdot)$ in $M_n(X)$ such that $J(x)$ is in Jordan form for each $x$, and $J(\cdot)$ is quasi-similar to $A(\cdot)$ in $M_n(X)$.*

**5. An application to $M_n(\mathscr{A})$.** If $\mathscr{A}$ is an abelian von Neumann algebra acting on the Hilbert space $\mathfrak{H}$, then the maximal ideal space $X$ of $\mathscr{A}$ is Stonian and the Gelfand map $\Gamma: \mathscr{A}\to C(X)$ is a *-isometrical isomorphism. Let $M_n(\mathscr{A})$ denote the von Neumann algebra consiting of all $n\times n$ matrices with entries in $\mathscr{A}$ acting on $\mathfrak{H}^n$ in the usual fashion. To each $A=(A_{ij})_{i,j=1}^n$ in $M_n(\mathscr{A})$ there corresponds a natural element $A(\cdot)$ in $M_n(X)$,

$$A(\cdot) = (\Gamma(A_{ij}))_{i,j=1}^n.$$

This correspondence is clearly a *-isomorphism.

If $R$ is an operator in $M_n(\mathscr{A})$, then the kernel of $R$ is the kernel of $R^*R$ and the projection $P$ onto the kernel of $R$ is a spectral projection for $R^*R$ and thus lies in the von Neumann algebra $M_n(\mathscr{A})$. Therefore if the kernel of $R$ is larger than $\{0\}$, there is a non-zero operator $P$ in $M_n(\mathscr{A})$ satisfying $RP=0$. By taking adjoints, one sees that if the range of $R$ is not dense, there is a non-zero $P$ in $M_n(\mathscr{A})$ such that $PR=0$. It follows that an operator $R$ in $M_n(\mathscr{A})$ has has zero kernel and dense range if and only if the corresponding element $R(\cdot)$ of $M_n(X)$ is quasi-invertible in $M_n(X)$. Therefore, if $A(\cdot)$ and $B(\cdot)$ are quasi-similar matrices in $M_n(X)$, then $A$ and $B$ are quasi-similar as operators on $\mathfrak{H}^n$. This observation yields the following theorem.

Theorem 5. 1. *If $\mathscr{A}$ is an abelian von Neumann algebra and if $A$ is an operator in $M_n(\mathscr{A})$, then there is an operator $J$ in $M_n(\mathscr{A})$ which is in Jordan form and is quasi-similar to $A$. That is,*

$$J = \begin{pmatrix} J_1 & P_1 & & & \\ & J_2 & P_2 & & \\ & & \cdot & \cdot & \\ & & & \cdot & P_{n-1} \\ & & & & J_n \end{pmatrix},$$

*where $J_i$ $(1 \leqq i \leqq n)$ and $P_j$ $(1 \leqq j \leqq n-1)$ are in $\mathscr{A}$ and the operators $P_j$ are projections.*

In [6], S. R. Foguel obtains a similar result using measure theoretic techniques.

A matrix of complex numbers which is in Jordan form can be written in an obvious way as the sum of a diagonal matrix and a nilpotent matrix. A simple calculation shows that the diagonal part and the nilpotent part commute. This observation has its obvious analog for matrices in $M_n(X)$. Using this analog, and the relationship between $M_n(\mathscr{A})$ and $M_n(X)$, the following is obtained:

Corollary 5. 2. *Every operator $A$ in $M_n(\mathscr{A})$ is quasi-similar to an operator $D+N$ where $D$ and $N$ are commuting operators in $M_n(\mathscr{A})$, $D$ is normal, and $N$ is nilpotent.*

We are now in a position to prove the basic theorem of this paper.

Theorem 5. 3. *Every non-scalar $n$-normal operator $A$ on a Hilbert space $\mathfrak{H}$ has a hyperinvariant subspace.*

Proof. By virtue of Theorem 2. 1 and Corollary 5. 2, we may assume that $A = D+N$, where $D$ is a normal operator, $N$ is a nilpotent operator, and $D$ and $N$ commute. In [5], Dunford shows that such operators are spectral operators, and hence if $T$ is an operator which commutes with $A$, then $T$ commutes with the resolution of the identity for $A$; that is, $T$ commutes with the spectral projections for $D$. Therefore, if $D$ is not a multiple of the identity operator, $D$ has spectral projections distinct from $0$ and $I$, and the ranges of the projections are hyperinvariant subspaces for $A$.

In case $D$ is scalar, then a subspace $\mathfrak{M}$ of $\mathfrak{H}$ will be hyperinvariant for $A$ just in case it is hyperinvariant for $N$. But $A$ is non-scalar, so $N$ cannot be the zero operator. On the other hand, $N$ is nilpotent so $N(\mathfrak{H})$ is not dense in $\mathfrak{H}$. Therefore $\mathfrak{M}=\overline{N(\mathfrak{H})}$ is a hyperinvariant subspace for $N$ and hence for $A$.

A simple argument extends Theorem 5. 3 to direct sums of $n$-normal operators. If $\alpha$ is a non-zero scalar, then the scalar matrices

$$\begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda & \alpha & & & \\ & \lambda & \alpha & & \\ & & \ddots & \ddots & \\ & & & \ddots & \alpha \\ & & & & \lambda \end{pmatrix}$$

are similar. In fact, the former is the Jordan form for the latter. Thus if $J$ is a scalar matrix in Jordan form, and $J$ is written as $D+N$ where $D$ is a diagonal matrix and $N$ has zero entries in every position except perhaps on the diagonal above the main diagonal where $N$ may have some ones, then $J$ is similar to $D+\alpha N$. Now if $X$ is a Stonian space, and $J(\cdot)$ as an element of $M_n(X)$ is pointwise in Jordan form, then by writing $J(\cdot) = D(\cdot)+N(\cdot)$ as in the scalar case and by using Theorem 3. 2, we get that $J(\cdot)$ is quasi-similar in $M_n(X)$ to $D(\cdot)+\alpha N(\cdot)$. Thus, using the representation of $M_n(\mathcal{A})$ as $M_n(X)$, we see that in Corollary 5. 2 if $N$ is not zero then we can arrange things so that the norm of $N$ is any positive number.

Theorem 5. 4. *If for each integer $i\geqq 0$, $A_i$ is a possibly zero i-normal operator acting on the Hilbert space $\mathfrak{H}_i$, and if $A$ is the direct sum operator $\Sigma\oplus A_i$, then if $A$ is non-scalar $A$ has a hyperinvariant subspace.*

Proof. Each $A_i$ is quasi-similar to an operator $D_i+N_i$ where $D_i$ and $N_i$ commute, $D_i$ is normal, $N_i$ is nilpotent of index at most $i$ and $\|N_i\|\leqq 1/i$. Thus $A$ is quasi-similiar to $D+N$ where $D = \Sigma\oplus D_i$ and $N = \Sigma\oplus N_i$. Clearly, $D$ is normal and commutes with $N$. Furthermore,

$$\|N^n\| = \sup_i \|N_i^n\| = \sup_{i>n} (1/i)^n \leqq (1/n)^n.$$

Therefore $\|N^n\|^{1/n}$ converges to zero, $N$ is quasinilpotent, and $D+N$ is a spectral operator. If $D$ is not a multiple of the identity operator, then the spectral subspaces for $D$ are hyperinvariant for $D+N$. If $D$ is a scalar operator, then $N$, as a non-zero direct sum of nilpotent operators, will have many hyperinvariant subspaces and these will be hyperinvariant for $D+N$ also. In any case, $D+N$ has hyperinvariant subspaces, and, by Theorem 2. 1 so does $A$.

Translated into the language of von Neumann algebras, Theorem 5. 4 says that every operator which belongs to a type I finite von Neumann algebra has a hyperinvariant subspace.

**6.** The term $n$-normal has been used with a somewhat broader meaning than that which we have given it. The algebras $M_n(\mathscr{A})$, where $\mathscr{A}$ is an abelian von Neumann algebra, are commonly said to be of type $\mathrm{I}_n$, and a von Neumann algebra is $n$-normal if it is the direct sum of algebras of type $\mathrm{I}_k$ where $k \leqq n$. According to [8] an operator is $n$-normal if the von Neumann algebra it generates is $n$-normal. To avoid confusion, we will say that operators which are $n$-normal in this latter sense are of *type n.*

In [7], for example, a von Neumann algebra $\mathscr{V}$ is equivalently defined as $n$-normal if it is satisfies the identity $\Sigma \pm X_{i_1} X_{i_2} \dots X_{i_{2n}} = O$, where $X_i \, (i=1, \dots, 2n)$ are arbitrary elements of $\mathscr{V}$, the sum is taken over all permutations of $(1, 2, \dots, 2n)$, and the sign is determined by the parity of the permutation. This characterization makes it clear that any von Neumann subalgebra of an $n$-normal algebra is $n$-normal, and thus that an operator which is $n$-normal in our sense is of type $n$. That the converse is false can be seen in the following example.

Let $\mathscr{A}$ denote the multiplication algebra acting on $L^2$ of the unit circle with Lebesgue measure, and let $\mathscr{C}$ be the algebra of all operators on one-dimensional Hilbert space. If $\mathscr{V}$ denotes the direct sum algebra $\mathscr{C} + M_2(\mathscr{A})$ and if $T$ is the operator

$$I \oplus \begin{pmatrix} -1 & S \\ O & O \end{pmatrix}$$

where $S$ is multiplication by the coordinate function, then $T$ generates $\mathscr{V}$, and thus $T$ is of type 2. But $T$ is not $n$-normal for any $n$; for suppose it were. Then $T$ and hence $\mathscr{V}$ are contained in some $M_n(\mathscr{W})$ where $\mathscr{W}$ is an abelian von Neumann algebra, and, since $T$ is not normal, $n$ is at least 2. Next consider the rank one projection $P$ which is the direct sum of the identity element of $C$ and the zero element of $M_2(\mathscr{A})$. By Theorem 1 of [8], $P$ is unitarily equivalent to a diagonal element $D$ of $M_n(\mathscr{W})$. If $D_1, \dots, D_n$ are the diagonal entries in $D$, then for some $j$, $1 \leqq j \leqq n$, $D_j$ is a rank one projection in $\mathscr{W}$, and the diagonal operator $E$, all of whose main diagonal entries are equal to $D_j$, is a rank $n$ projection which commutes with $M_n(\mathscr{W})$ and hence with $\mathscr{V}$. On the other hand, the commutant of $\mathscr{V}$ consists of all operators of the form

$$\lambda I \oplus \begin{pmatrix} A & O \\ O & A \end{pmatrix}$$

where $\lambda$ is any scalar and $A$ is in $\mathscr{A}$. In the particular, if the commuting operator is a projection, then $\lambda$ is 0 or 1 and $A$ is multiplication by the characteristic function of some measurable set. The only way such a projection can be of finite rank is for $A$ to be the zero operator, and so the only non-zero, finite rank projection which commutes with the algebra $\mathscr{V}$ is the rank one projection $P$. But $E$ is a projection of rank $n \neq 1$ and $E$ commutes with $\mathscr{V}$. This is a contradiction and thus the original assumption, that $T$ was $n$-normal, is false.

Fortunately, this confusion over definitions causes no problems with our Theorem 5. 3, for if $T$ is a non-scalar operator of type $n$, then $T$ is a direct sum of operators which are $i$-normal for some $i$ and hence, by Theorem 5. 4, $T$ has a hyperinvariant subspace.

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